Nonlinear Bogolyubov–Valatin transformations and quaternions

J-W van Holten[1,2] **and K Scharnhorst**[2]

[1] NIKHEF, PO Box 41882, 1009 DB Amsterdam, The Netherlands
[2] Department of Theoretical Physics, Division of Physics and Astronomy,
Faculty of Exact Sciences, Vrije Universiteit Amsterdam, De Boelelaan 1081,
1081 HV Amsterdam, The Netherlands

E-mail: v.holten@nikhef.nl and scharnh@nat.vu.nl

Abstract
In introducing second quantization for fermions, Jordan and Wigner (1927, 1928) observed that the algebra of a single pair of fermion creation and annihilation operators in quantum mechanics is closely related to the algebra of quaternions **H**. For the first time, here we exploit this fact to study nonlinear Bogolyubov–Valatin transformations (canonical transformations for fermions) for a single fermionic mode. By means of these transformations, a class of fermionic Hamiltonians in an external field is related to the standard Fermi oscillator.

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Unitary transformations play a prominent role in quantum mechanics. Like canonical transformations in classical mechanics, unitary transformations of quantum dynamical degrees of freedom often simplify the dynamical equations, or allow us to introduce sensible approximation schemes. Such methods have wide-ranging applications, from the study of simple systems to many-body problems in solid-state or nuclear physics and quantum chemistry, up to the infinite-dimensional systems of quantum field theory [1–4]. Linear (unitary) canonical transformations (i.e., transformations preserving the canonical anticommutation relations (CAR)) for fermions have been introduced by Bogolyubov and Valatin (for two fermionic modes) in connection with the study of the mechanism of superconductivity [5–9]. These (linear) Bogolyubov–Valatin transformations have been extended, initially by Bogolyubov and his collaborators [10, 11], [12, appendix II, p 123] ([13, p 116], [14, p 679]), to involve *n* fermionic modes (so-called generalized linear Bogolyubov–Valatin transformations, see, e.g., [15, Part III, p 247] ([16, p 341])). Such linear canonical transformations are important from a physical as well as from a mathematical

point of view. Mathematically, they allow us to relate quite arbitrary Hamiltonians, quadratic in the fermion creation and annihilation operators, to collections of Fermi oscillators whose mathematics is very well understood. From a physical point of view, canonical transformations implement the concept of quasiparticles in terms of which the physical processes taking place can be described and understood in an effective and transparent manner. To apply the powerful tool of canonical transformations to the physically interesting class of non-quadratic Hamiltonians, however, requires us to go beyond linear Bogolyubov–Valatin transformations. Certain aspects of nonlinear Bogolyubov–Valatin transformations have received some attention over time [17–36]. (We disregard here work done within the framework of the coupled-cluster method (CCM) [4] which is nonunitary.) However, a systematic analytic study of general (nonlinear) Bogolyubov–Valatin transformations has not been undertaken so far. In the present paper, as a first step towards this goal we are going to investigate the prototypical case of a single fermionic mode.

Let us consider a pair of fermion creation and annihilation operators $\hat{a}^+, \hat{a}$. Here, we regard the creation operator $\hat{a}^+$ as the Hermitian conjugate of the annihilation operator $\hat{a}$: $\hat{a}^+ = \hat{a}^\dagger$ (we will use the latter notation throughout). They obey the CAR

$$\{\hat{a}^\dagger, \hat{a}\} = \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger = 1, \tag{1}$$

$$(\hat{a}^\dagger)^2 = \hat{a}^2 = 0. \tag{2}$$

It is now instructive to consider the following pair of anti-Hermitian operators:

$$\hat{a}^{[1]} = -\hat{a}^{[1]\dagger} = i(\hat{a} + \hat{a}^\dagger), \tag{3}$$

$$\hat{a}^{[2]} = -\hat{a}^{[2]\dagger} = \hat{a} - \hat{a}^\dagger. \tag{4}$$

These two operators obey the equation ($p, q = 1, 2$)

$$\{\hat{a}^{[p]}, \hat{a}^{[q]}\} = -2\delta_{pq}. \tag{5}$$

Consequently, they generate the (real) Clifford algebra $C(0, 2)$ which is isomorphic to the algebra of quaternions **H** (cf, e.g., [37, chapter 15, p 123], [38, chapter 16, p 205]). We can define the three quaternionic units **i**, **j**, **k** by the equations

$$\mathbf{i} = \hat{a}^{[1]} = i(\hat{a} + \hat{a}^\dagger), \tag{6}$$

$$\mathbf{j} = \hat{a}^{[2]} = \hat{a} - \hat{a}^\dagger, \tag{7}$$

$$\mathbf{k} = \hat{a}^{[3]} = \hat{a}^{[1]}\hat{a}^{[2]} = i(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger). \tag{8}$$

Quite generally, these definitions entail that any pair of fermionic creation and annihilation operators $\hat{a}^\dagger, \hat{a}$ induces a (bi-)quaternionic structure into any consideration and model they are a part of. And in turn, any quaternionic structure can be interpreted in terms of fermionic creation and annihilation operators. The link between the algebra of quaternions **H** and fermion creation and annihilation operators has been mentioned for the first time by Jordan and Wigner in introducing second quantization for fermions [39, p 474], [40, p 635] ([41, p 45], [42, p 113]). However, it seems to not have found its way into the work of later authors. (The only further mention of this fact in the literature we have been able to find is in [43].)

Let us now start by writing down an ansatz for the most general Bogolyubov–Valatin transformation for a single fermionic mode. In view of equations (1) and (2), the new pair of fermion annihilation and creation operators $\hat{b}, \hat{b}^\dagger$ reads (here we assume the coefficients to be complex numbers: $\lambda^{(k;l)} \in \mathbf{C}, k, l = 0, 1; \{\lambda\} = \{\lambda^{(0;0)}, \lambda^{(0;1)}, \lambda^{(1;0)}, \lambda^{(1;1)}\}$)

$$\hat{b} = \mathsf{B}(\{\lambda\}; \hat{a}) = \lambda^{(0;0)} + \lambda^{(0;1)}\hat{a} + \lambda^{(1;0)}\hat{a}^\dagger + \lambda^{(1;1)}\hat{a}^\dagger\hat{a}, \tag{9}$$

$$\hat{b}^\dagger = \mathsf{B}(\{\lambda\}; \hat{a})^\dagger = \overline{\lambda^{(0;0)}} + \overline{\lambda^{(0;1)}}\hat{a}^\dagger + \overline{\lambda^{(1;0)}}\hat{a} + \overline{\lambda^{(1;1)}}\hat{a}^\dagger\hat{a}. \tag{10}$$

From equation (1) applied to $\hat{b}, \hat{b}^\dagger$ follows:

$$2|\lambda^{(0;0)}|^2 + |\lambda^{(1;0)}|^2 + |\lambda^{(0;1)}|^2 = 1 \tag{11}$$

and from equation (2) follows:

$$(\lambda^{(0;0)})^2 + \lambda^{(1;0)}\lambda^{(0;1)} = 0, \tag{12}$$

while both equations (1) and (2) also yield

$$2\lambda^{(0;0)} + \lambda^{(1;1)} = 0 \tag{13}$$

([31, section 2.6, p 32, equation (2.91)]). Using equation (12), equation (11) can be transformed to read

$$|\lambda^{(1;0)}| + |\lambda^{(0;1)}| = 1. \tag{14}$$

(Take absolute values on both sides of the modified equation (12): $(\lambda^{(0;0)})^2 = -\lambda^{(1;0)}\lambda^{(0;1)}$, eliminate $|\lambda^{(0;0)}|^2$ from equation (11) and take the square root.) For comparison, let us have a look at the class of generalized linear Bogolyubov–Valatin transformations (for one mode!): $\lambda^{(0;0)} = \lambda^{(1;1)} = 0$. Then, equation (12) requires that $\lambda^{(1;0)}\lambda^{(0;1)} = 0$. This condition allows two solutions:

$$\lambda^{(1;0)} = 0, \qquad |\lambda^{(0;1)}| = 1, \tag{15}$$

$$\lambda^{(0;1)} = 0, \qquad |\lambda^{(1;0)}| = 1. \tag{16}$$

It has been found that generalized linear Bogolyubov–Valatin transformations (for n modes) are equivalent to the group of $O(2n, \mathbf{R})$ transformations which is in accord (for $n = 1$) with equations (15) and (16). (This group is reduced to $SO(2n, \mathbf{R})$ if one only allows transformations continuously connected to the identity map—then in our case only equation (15) applies [44–49], [23, 50], [51, section 3.2, p 16], [1, section 2.2, p 38], [52, section 9.1, p 111], [53, section 9.1, p 127]; if one does not assume that $\hat{a}$ and $\hat{a}^+$ are Hermitian conjugates of each other, the corresponding groups are $O(2n, \mathbf{C})$ and $SO(2n, \mathbf{C})$, respectively [47, 48, 54, 55], [1, section 2.1, p 34], [56, 57].)

The Bogolyubov–Valatin transformation (9) can be inverted. We can write

$$\hat{a} = \mathsf{B}(\{\nu\}; \hat{b}) = \nu^{(0;0)} + \nu^{(0;1)}\hat{b} + \nu^{(1;0)}\hat{b}^\dagger + \nu^{(1;1)}\hat{b}^\dagger\hat{b}. \tag{17}$$

Inserting equation (9) into equation (17), one obtains a system of linear equations in $\{\nu\}$ whose (unique) solution reads

$$\nu^{(0;0)} = \overline{\lambda^{(0;0)}}\lambda^{(1;0)} - \lambda^{(0;0)}\overline{\lambda^{(0;1)}}, \tag{18}$$

$$\nu^{(0;1)} = \overline{\lambda^{(0;1)}}, \tag{19}$$

$$\nu^{(1;0)} = \lambda^{(1;0)}, \tag{20}$$

$$\nu^{(1;1)} = -2\nu^{(0;0)}. \tag{21}$$

One can convince oneself by explicit calculation that $\{\nu\}$ given by equations (18)–(21) obey the analogues of equations (11) and (12) if $\{\lambda\}$ obey the latter equations. Furthermore, the nonlinear Bogolyubov–Valatin transformations (9) form a group G_{BV}. After the above considerations, it remains to check that $\mathsf{B}(\{\nu\}; \hat{a}) = \mathsf{B}(\{\mu\}; \mathsf{B}(\{\lambda\}; \hat{a}))$ belongs to G_{BV} if $\mathsf{B}(\{\lambda\}; \hat{a})$ and $\mathsf{B}(\{\mu\}; \hat{a})$ belong to G_{BV}. One can explicitly check that $\{\nu\}$ obey the analogues of equations (11) and (12) if $\{\lambda\}, \{\mu\}$ obey equations (11) and (12), or their analogues, respectively.

To further study the Bogolyubov–Valatin group G_{BV}, it now turns out to be useful to consider the linear vector space V generated by the operators $\hat{a}, \hat{a}^\dagger$ (V is the space of

linear operators in Fock space). It is four dimensional and is spanned by the operator basis $a^T = (1, \hat{a}, \hat{a}^\dagger, \hat{a}^\dagger\hat{a})$. However, taking into account the connection already discussed between the operators $\hat{a}, \hat{a}^\dagger$ and quaternions, it turns out to be advantageous to pursue the consideration of this linear space in terms of the operator basis (cf equations (6)–(8)) $a^T = (1, \hat{a}^{[1]}, \hat{a}^{[2]}, \hat{a}^{[3]})$. The Bogolyubov–Valatin transformation (9) can be understood as a base transformation in the linear space V. We can write ($b^T = (1, \hat{b}^{[1]}, \hat{b}^{[2]}, \hat{b}^{[3]})$)

$$b = \mathcal{A}(\{\lambda\})a, \tag{22}$$

where the 4×4 matrix $\mathcal{A}(\{\lambda\})$ is a block diagonal matrix $\mathcal{A} = \mathrm{diag}(1, A)$ and $A = A(\{\lambda\})$ is the real 3×3 matrix

$$A(\{\lambda\}) = \begin{pmatrix} \mathrm{Re}\,\kappa^{(0;1)} & \mathrm{Re}\,\kappa^{(1;0)} & \mathrm{Re}\,\kappa^{(1;1)} \\ \mathrm{Im}\,\kappa^{(0;1)} & \mathrm{Im}\,\kappa^{(1;0)} & \mathrm{Im}\,\kappa^{(1;1)} \\ \mathrm{Im}(\overline{\kappa^{(1;0)}}\,\kappa^{(1;1)}) & \mathrm{Im}(\overline{\kappa^{(1;1)}}\,\kappa^{(0;1)}) & \mathrm{Im}(\overline{\kappa^{(0;1)}}\,\kappa^{(1;0)}) \end{pmatrix} \tag{23}$$

with (taking into account equations (11)–(13)) unit determinant ($\det A = 1$) and inverse $A(\{\lambda\})^{-1} = A(\{\lambda\})^T$. Here, we have applied the notation

$$\kappa^{(0;1)} = \lambda^{(0;1)} + \lambda^{(1;0)}, \tag{24}$$

$$\kappa^{(1;0)} = \mathrm{i}(\lambda^{(0;1)} - \lambda^{(1;0)}), \tag{25}$$

$$\kappa^{(1;1)} = \lambda^{(1;1)} = -2\lambda^{(0;0)} = -2\kappa^{(0;0)}. \tag{26}$$

In view of the above considerations, the Bogolyubov–Valatin group G_{BV} is equivalent to the group $SO(3)$. Given the link between creation and annihilation operators and the algebra of quaternions $\mathbf{H}$ discussed further above, this does not come as a big surprise. In accordance with equations (6)–(8), the new pair of operators $\hat{b}, \hat{b}^\dagger$ defines a transformed system of quaternionic units $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ by writing

$$\mathbf{i}' = \hat{b}^{[1]} = \mathrm{i}(\hat{b} + \hat{b}^\dagger) = \mathrm{Re}\,\kappa^{(0;1)}\mathbf{i} + \mathrm{Re}\,\kappa^{(1;0)}\mathbf{j} + \mathrm{Re}\,\kappa^{(1;1)}\mathbf{k}, \tag{27}$$

$$\mathbf{j}' = \hat{b}^{[2]} = \hat{b} - \hat{b}^\dagger = \mathrm{Im}\,\kappa^{(0;1)}\mathbf{i} + \mathrm{Im}\,\kappa^{(1;0)}\mathbf{j} + \mathrm{Im}\,\kappa^{(1;1)}\mathbf{k}, \tag{28}$$

$$\mathbf{k}' = \hat{b}^{[3]} = \hat{b}^{[1]}\hat{b}^{[2]}. \tag{29}$$

In terms of the new parameters $\{\kappa\}$ (equations (24)–(26)), equations (11) and (12) read

$$|\kappa^{(0;1)}|^2 + |\kappa^{(1;0)}|^2 + |\kappa^{(1;1)}|^2 = 2, \tag{30}$$

$$(\kappa^{(0;1)})^2 + (\kappa^{(1;0)})^2 + (\kappa^{(1;1)})^2 = 0. \tag{31}$$

Let us now further analyse these equations. Separating them into real and imaginary parts and introducing the three-dimensional (complex) vector $(\mathbf{e}')^T = (\kappa^{(0;1)}, \kappa^{(1;0)}, \kappa^{(1;1)})$, these (three real) equations can compactly be written as

$$(\mathbf{e}')^T\mathbf{e}' = 0, \qquad |\mathbf{e}'|^2 = 2. \tag{32}$$

$\mathbf{e}'$ is an *isotropic vector* (cf, e.g., [58, section 6.3, p 113], and [59] for some more detailed and pedagogical exposition). In a way, it appears to be an interesting feature that within the framework of general (nonlinear) Bogolyubov–Valatin transformations for a single pair of fermion creation and annihilation operators, spinors make their appearance (via isotropic vectors, cf, e.g., [59]). The properties of these spinors are related to canonical (Bogolyubov–Valatin) transformations. Introducing two three-dimensional (real) vectors $\mathbf{e}'_1 = \mathrm{Re}(\mathbf{e}')$, $\mathbf{e}'_2 = \mathrm{Im}(\mathbf{e}')$ (transposed, they agree with the first two rows of the matrix (23)), one can write equations (32) as

$$|\mathbf{e}'_1|^2 = |\mathbf{e}'_2|^2 = 1, \qquad (\mathbf{e}'_1)^T\mathbf{e}'_2 = 0. \tag{33}$$

These equations define the vectors $\mathbf{e}_1'$, $\mathbf{e}_2'$ as a pair of orthonormal vectors which can be supplemented by the vector $\mathbf{e}_3' = \mathbf{e}_1' \times \mathbf{e}_2'$ to form an orthonormal vector triple in $\mathbf{R}_3$. It is worth mentioning here that the vector $(\mathbf{e}_3')^T$ coincides with the third row of the matrix (23). Consequently, the orthogonality condition(s) for the matrix (23) are equivalent to the conditions for the Bogolyubov–Valatin transformation to be canonical (equations (30), (31) or (11), (12)). This is a generalization of an insight obtained for linear Bogolyubov–Valatin transformation (see [44, 47, 49]) to the general (nonlinear) case.

The canonical (Bogolyubov–Valatin) transformation (9) can be implemented by means of an unitary transformation $\mathsf{U}(\{\lambda\}; \hat{a})$:

$$\hat{b} = \mathsf{B}(\{\lambda\}; \hat{a}) = \mathsf{U}(\{\lambda\}; \hat{a})\hat{a}\,\mathsf{U}(\{\lambda\}; \hat{a})^{\dagger}. \tag{34}$$

The analogue of equation (34)

$$\hat{b}^{[1]} = \mathsf{U}(\{\lambda\}; \hat{a})\hat{a}^{[1]}\,\mathsf{U}(\{\lambda\}; \hat{a})^{\dagger} \tag{35}$$

has a remarkable interpretation in terms of quaternions discussed further above (an analogous comment applies to $\hat{a}^{[2]}$ and $\hat{a}^{[3]}$). Equations (27) and (35) are just concrete realizations of the theory of rotations in the language of quaternions first elaborated by Cayley and Hamilton (cf, e.g., [58, section 12.8, p 215], equation (9), [60, section 4.5, p 201]). Equation (27) represents a ($SO(3)$) rotation of the vector $(1, 0, 0)$ in the three-dimensional space spanned by the quaternionic units $\mathbf{i}, \mathbf{j}, \mathbf{k}$, while equation (35) stands for the corresponding ($SU(2)$) transformation of the quaternion $\mathbf{i}$ ($= \hat{a}^{[1]}$) by quaternionic multiplication. The unitary operator $\mathsf{U}(\{\lambda\}; \hat{a})$ can be understood as a unit quaternion given by ($-\pi < \phi \leqslant \pi, n_1, n_2, n_3 \in \mathbf{R}$, $n^2 = 1$)

$$\mathsf{U}(\{\lambda\}; \hat{a}) = \cos\frac{\phi}{2} + \sin\frac{\phi}{2}(n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}) \tag{36}$$

$$= \exp(\phi(n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k})/2). \tag{37}$$

The coefficients $\{\lambda\}$ are given in terms of the parameters ϕ, n_1, n_2, n_3 by the equations

$$\lambda^{(0;1)} = \left(\cos\frac{\phi}{2} - \mathrm{i}\,n_3\sin\frac{\phi}{2}\right)^2, \tag{38}$$

$$\lambda^{(1;0)} = (n_1 + \mathrm{i}\,n_2)^2\sin^2\frac{\phi}{2}, \tag{39}$$

$$\lambda^{(1;1)} = -2\lambda^{(0;0)} = 2\mathrm{i}\left(\cos\frac{\phi}{2} - \mathrm{i}\,n_3\sin\frac{\phi}{2}\right)(n_1 + \mathrm{i}\,n_2)\sin\frac{\phi}{2}. \tag{40}$$

To obtain these relations, insert equation (36) into equation (34) and compare the rhs with equation (9). From the representation (37), one sees immediately that the operators $\mathbf{i} = \hat{a}^{[1]}, \mathbf{j} = \hat{a}^{[2]}, \mathbf{k} = \hat{a}^{[3]}$ (cf equations (6)–(8)) are generators of the group $SU(2)$ and they obey the Lie algebra of $SO(3)$, $SU(2)$. This has been observed earlier (in a more general context) in [61] (also see [19]). Related observations can be found in [62, appendix A.1, p 919], [63] and [64, p 907], equation (6.2). One can convince oneself that for linear Bogolyubov–Valatin transformations of type (15), equation (37) agrees (sometimes up to some elementary complex phase factor) with equation (7) in [65], with equation (5.1) in [49], with equation (3.6) in [55], with equation (2.32a), p 40, section 2.2, in [1] and with equation (3.10) in [66] (reduced to the one-mode case; incidentally, there is disagreement with [67, p 205], below of equation (11)).

The vacuum state $|0\rangle$ defined by $\hat{a}|0\rangle = 0$ transforms under (general, i.e., $SO(3)$) Bogolyubov–Valatin transformations according to the law

$$|0\rangle_{\{\lambda\}} = \mathsf{U}(\{\lambda\}; \hat{a})|0\rangle, \qquad \hat{b}|0\rangle_{\{\lambda\}} = 0. \tag{41}$$

Associating $|0\rangle$ with a vector in a two-dimensional (complex) Hilbert space and $\mathsf{U}(\{\lambda\}; \hat{a})$ with a 2×2 matrix operating in it (cf [39, pp 474–5], [40, p 634] ([41, p 44], [42, p 112])), one sees that this vector transforms as a spinor (with a corresponding element of $SU(2)$) under Bogolyubov–Valatin transformations. The state $|0\rangle_{\{\lambda\}}$ is a spin ($SU(2)$) coherent state [23] (with respect to the $\hat{a}, \hat{a}^\dagger$ operators, cf, e.g., [68, section I.4, p 25], [52, section 4.3, p 59], [53, section 4.3, p 72], [64, section III.D.1, p 884 and section VI.A.1, p 907]). However, these fermion coherent states are different (cf the comments in [68, section I.5, p 55] and in [64, section VI.D, p 919]) from the Grassmann (fermion) coherent states (see, e.g., [68, section I.5, p 48]).

Finally, let us have a look at the standard Fermi oscillator given by the Hamiltonian $H = \hat{a}^\dagger \hat{a} - \frac{1}{2}$. Applying the Bogolyubov–Valatin transformation (9), one can see that it is unitarily equivalent to the following class of fermionic oscillators in an external field [17–19, 22, 26], [31, section 2.6, p 29] (below, we have taken into account equations (11)–(14)):

$$
\begin{aligned}
H' = \hat{b}^\dagger \hat{b} - \tfrac{1}{2} &= \mathsf{U}(\{\lambda\}; \hat{a}) H\, \mathsf{U}(\{\lambda\}; \hat{a})^\dagger \\
&= (|\lambda^{(0;1)}| - |\lambda^{(1;0)}|)\big(\hat{a}^\dagger \hat{a} - \tfrac{1}{2}\big) + (\overline{\lambda^{(0;0)}} \lambda^{(0;1)} - \lambda^{(0;0)} \overline{\lambda^{(1;0)}})\hat{a} \\
&\quad + (\lambda^{(0;0)} \overline{\lambda^{(0;1)}} - \overline{\lambda^{(0;0)}} \lambda^{(1;0)})\hat{a}^\dagger.
\end{aligned}
\tag{42}
$$

As a special case, equation (42) contains for $|\lambda^{(0;1)}| = |\lambda^{(1;0)}|$ also Hamiltonians that are linear in the creation and annihilation operators (such Hamiltonians have been studied in [22, section 4, p 477]). Equation (42) demonstrates that any (Hermitian) Hamiltonian H_0 ($0 \leqslant \alpha \in \mathbf{R}, \beta \in \mathbf{C}$)

$$
H_0 = \alpha\big(\hat{a}^\dagger \hat{a} - \tfrac{1}{2}\big) + \beta \hat{a} + \overline{\beta} \hat{a}^\dagger
\tag{43}
$$

can be written and understood in terms of transformed creation and annihilation operators $\hat{b}^\dagger, \hat{b}$ as

$$
H_0 = \sqrt{\alpha^2 + 4|\beta|^2}\big(\hat{b}^\dagger \hat{b} - \tfrac{1}{2}\big).
\tag{44}
$$

In view of the above considerations, its dynamical (spectrum generating) algebra is $so(3) \sim su(2)$.

The present paper paves the way for the study of (nonlinear) Bogolyubov–Valatin transformations in full generality for any finite number of fermionic modes. This is done by introducing a methodological framework which can be generalized (stepwise) to more than just one mode. For several—say n—fermionic modes, the formalism can, for example, be expected to allow equivalences of wide classes of non-quadratic fermionic Hamiltonians to collections of n Fermi oscillators to be derived. This will be of considerable interest for a wide range of physically relevant models. However, beyond its plain methodological value, the present study of nonlinear Bogolyubov–Valatin transformations for just one fermionic mode provides us even with some surprising insight. Note that the nonlinear Bogolyubov–Valatin transformation (9) defines fermionic operators as a sum of Fermi-even and Fermi-odd terms. This is reminiscent of a supersymmetric transformation (cf in this respect [36]). In this context, remember that linear Bogolyubov–Valatin transformations—in contrast to (bosonic) linear Bogolyubov transformations—do not allow any linear shifts by complex numbers to be performed (see, e.g., [69, appendix IV, section 1(a), 1st edn, p 280, 2nd edn, p 292] ([70, p 328]), also see [1, section 2.4, p 40], equation (2.33*b*)).

The future generalization of the present work to several fermionic modes can alternatively also be understood as generalizing the complex coefficients $\{\lambda\}$ in equation (9) to operator-valued functions for which the present analysis has to be repeated in an appropriately modified manner. Closely related to this direction of future research is to consider the coefficients $\{\lambda\}$ as elements of an appropriately chosen Grassmann algebra. However, the coefficients $\{\lambda\}$ can not only be imagined to be functions of fermionic operators but also to be functions of bosonic

creation and annihilation operators. Such constructions are met, for example, in the study of so-called quantized Bogolyubov–Valatin transformations (introduced in [71]) and, more generally, in the study of boson–fermion interactions (see, e.g., [2, chapter 5, p 108]).

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